

Relativistic bound states at Born level*

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Theoretical and phenomenological studies indicate that the QCD coupling $\alpha_s(Q^2)$ freezes in the infrared. Hadrons may then be described by a perturbative expansion around “Born” states bound only by a confining potential. A linear potential results from the QCD equations of motion when Gauss’ law for A^0 is solved with $F_{\mu\nu}^a F_a^{\mu\nu} \neq 0$ as boundary condition. The $\mathcal{O}(\alpha_s^0)$ Born states are Poincaré covariant and can serve as $|in\rangle$ and $\langle out|$ states of scattering amplitudes. Their Dirac-type wave functions include $f\bar{f}$ creation/annihilation effects giving sea-like partons at low x_{Bj} .

1. Bound states at $\mathcal{O}(\alpha_s^0)$

Hadrons are highly relativistic bound states. The mass difference between excited states is of the same order as light hadron masses, which in turn are much larger than the u, d, s (current) quark masses. Parton distributions reveal the relativistic motion of quarks in the nucleon, and the presence of a non-vanishing sea quark distribution even at low scales [2].

Relativistic dynamics and color confinement are often thought to imply that the QCD coupling $\alpha_s(Q^2)$ is large at small momentum scales Q . Hadrons nevertheless have features which seem difficult to reconcile with a strongly coupled theory. To name a few:

- Hadron spectra reflect their valence quark ($q\bar{q}$ and qqq) degrees of freedom. There is no firm evidence for exotic, glueball or hybrid states. The sea quarks do not manifest themselves in the excitation spectrum.
- The OZI rule [3]. *E.g.*, the $\phi(1020)$ decays predominantly to $K\bar{K}$, even though this final state is barely allowed kinematically. In a strong

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coupling scenario one would expect little suppression of $s\bar{s} \leftrightarrow u\bar{u}, d\bar{d}$ transitions at the mass scale of the strange quark, $m_s \sim 100$ MeV.

- Perturbation theory explains many features of hadron production down to low momentum scales [4].

Considerations like the above motivate studying the possibility that α_s is of moderate size even in the confinement domain. This is not as heretical as it may sound. Several theoretical and phenomenological studies [5] concur that the strong coupling freezes at a moderate value. The quark model gives a semi-quantitative understanding of hadrons using the perturbative QCD potential added to a spin-independent linear potential. Features like the $\Sigma - \Lambda$ mass splitting are then explained by single gluon exchange [6].

How could the confining interaction be self-consistently described theoretically, and combined with perturbative QCD? One possibility is to impose a non-vanishing boundary condition on $F_{\mu\nu}^a F_a^{\mu\nu}$ in the solution of Gauss' law [7]. This can be illustrated in QED. Taking the diagonal matrix element of $-\nabla^2 A^0(\mathbf{x}) = e\psi^\dagger\psi(\mathbf{x})$, for a state where an electron is at \mathbf{x}_1 and a positron at \mathbf{x}_2 , gives $4\pi A^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2) = e/|\mathbf{x} - \mathbf{x}_1| - e/|\mathbf{x} - \mathbf{x}_2|$. The standard Coulomb potential is then $\frac{1}{2}[eA^0(\mathbf{x}_1) - eA^0(\mathbf{x}_2)] = -\alpha/|\mathbf{x}_1 - \mathbf{x}_2|$. If a non-vanishing field strength at spatial infinity is imposed,

$$\lim_{|\mathbf{x}| \rightarrow \infty} F_{\mu\nu}^a F_a^{\mu\nu}(\mathbf{x}) = -2\Lambda^4 \quad (1)$$

the solution of Gauss' law includes a homogeneous term,

$$A^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2) = \Lambda^2 \hat{\ell} \cdot \mathbf{x} + e/4\pi|\mathbf{x} - \mathbf{x}_1| - e/4\pi|\mathbf{x} - \mathbf{x}_2| \quad (2)$$

where the unit vector $\hat{\ell}(\mathbf{x}_1, \mathbf{x}_2)$ may depend on the positions of the electron and the positron but not on \mathbf{x} . Stationarity of the action wrt. variations in $\hat{\ell}$ sets $\hat{\ell} \parallel \mathbf{x}_1 - \mathbf{x}_2$. Thus arises an instantaneous confining interaction in the Hamiltonian [7],

$$H_\Lambda = -\frac{e\Lambda^2}{4} \int d\mathbf{x} d\mathbf{y} \psi^\dagger\psi(t, \mathbf{x})|\mathbf{x} - \mathbf{y}|\psi^\dagger\psi(t, \mathbf{y}) \quad (3)$$

Clearly we should set $\Lambda = 0$ in QED. However, a similar analysis can be carried out for QCD, where a linear potential is called for by data, lattice calculations and the quark model. The boundary condition (1) provides a dimensionful parameter which is not present in the Lagrangian, and which can be determined to describe (although not explain) confinement. Gauge covariant bound states exist for color singlet $q\bar{q}$ mesons and qqq baryons.

The linear potential in (3) is of $\mathcal{O}(e\Lambda^2)$ and thus leading compared to the $\mathcal{O}(e^2)$ perturbative potential. It was found to provide a Lorentz-covariant framework for bound states, giving energy eigenvalues with the correct dependence on the CM momentum [1, 8]. This is non-trivial for quantization at equal time, and indicates that the implementation of the non-vanishing boundary condition (1) preserves the Poincaré invariance.

The road thus appears open to a perturbative expansion. The bound states formed by the linear interaction potential in (3) take the place of the free $|in\rangle$ and $\langle out|$ states normally used in the scattering of pointlike particles. In effect, one perturbatively expands around “Born terms” which incorporate confinement but no perturbative interactions.

2. Wave functions

Relativistic dynamics necessarily involves pair creation and annihilation. This is manifest in the sea quark distribution of the proton, which persists down to low scales [2]. Consequently, relativistic bound states have an infinite number of Fock components. This need not exclude an analytic description, as demonstrated by the states of an electron in a static Coulomb field. The bound state energies E are given by the Dirac equation,

$$[-i\gamma^0\nabla\cdot\boldsymbol{\gamma} + eA^0(\mathbf{x}) + m\gamma^0]\phi(\mathbf{x}) = E\phi(\mathbf{x}) \quad (4)$$

which is obtained by summing all diagrams where the electron interacts with the external field. As was recognized early on in the “Klein paradox” [9], the Dirac wave function $\phi(\mathbf{x})$ includes e^+e^- pair effects. Time-ordering of the electron interactions shows that scattering into negative energy states corresponds to pair creation and annihilation.

The time-independence of $A^0(\mathbf{x})$ in (4) implies that the bound state energies E are unchanged if retarded (instead of Feynman) electron propagators are used in all diagrams¹. In retarded propagation only a single (positive or negative energy) electron is present at any time. The Dirac wave function $\phi(\mathbf{x})$ in (4) describes the electron with this boundary condition. In analogy to cross sections [10], retarded boundary conditions give *inclusive* rather than exclusive charge densities $\phi^\dagger\phi(\mathbf{x})$.

Electron pairs contribute significantly to the Dirac charge density whenever the potential is strong (and the dynamics thus is relativistic). Their contribution generally makes the Dirac wave function unnormalizable [11].

¹ A time-independent external field does not transmit energy. Hence the p^0 component of the electron momentum is preserved. If $p^0 > -m$ the negative-energy pole of the electron propagator at $p^0 = -\sqrt{\mathbf{p}^2 + m^2}$ is never probed. This makes the Green function $G(p^0, \mathbf{p})$ independent of the $i\varepsilon$ prescription at that pole [7].

This holds for any potential which is a polynomial in r or in $1/r$ – except for $V(r) \propto 1/r$. Similarly in $D = 1 + 1$ dimensions any potential that is a polynomial in x or $1/x$ gives unnormalizable wave functions. The absence of the normalization condition $\int d^3\mathbf{x} \phi^\dagger \phi(\mathbf{x}) = 1$ makes the Dirac energy spectrum *continuous*, quite unlike the discrete Schrödinger spectrum².

Fig. 1(a) shows the Dirac wave function in (4) for the QED₂ potential $eA^0(x) = \frac{1}{2}e^2|x|$. Since $m/e = 2.5$ the dynamics is nearly non-relativistic at low $|x|$, and close agreement with the corresponding solution of the Schrödinger equation is indeed found for $e|x| \lesssim 5$ if the Dirac solution is normalized to unity in this region. However, the Dirac wave function starts to oscillate when the potential reaches twice the electron mass, $e|x| \simeq 10$, indicative of contributions from e^+e^- pairs. Since $\phi(x \rightarrow \infty) \sim \exp(ie^2x^2/4)$ the Dirac charge density is asymptotically constant.

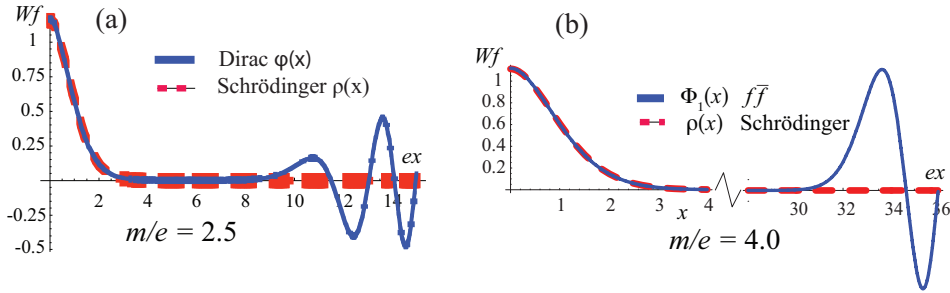


Fig. 1. Wave functions in $D = 1 + 1$ dimensions [1]. (a) Comparison of the upper component $\varphi(x)$ of the Dirac wave function (4) with the Schrödinger wave function $\rho(x)$ for $m/e = 2.5$. (b) Comparison of one component of the $f\bar{f}$ wave function (6) (for $m/e = 4.0$) with the Schrödinger wave function (for $m/e = 2.0$).

Retarded boundary conditions may plausibly be used with the instantaneous linear potential³ (3). Then the wave function $\Phi(\mathbf{x})$ of an $f\bar{f}$ bound state with CM momentum \mathbf{P} ,

$$|P, t\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_1(t, \mathbf{x}_1) \exp[i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2] \Phi(\mathbf{x}_1 - \mathbf{x}_2) \psi_2(t, \mathbf{x}_2) |0\rangle_R \quad (5)$$

satisfies (for $m_1 = m_2 = m$)

$$i\nabla_x \cdot \{\gamma^0 \gamma, \Phi(\mathbf{x})\} - \frac{1}{2}\mathbf{P} \cdot [\gamma^0 \gamma, \Phi(\mathbf{x})] + m [\gamma^0, \Phi(\mathbf{x})] = [E - V(\mathbf{x})] \Phi(\mathbf{x}) \quad (6)$$

where $V(\mathbf{x}) = \frac{1}{2}e\Lambda^2|\mathbf{x}|$. The wave function $\Phi(\mathbf{x})$ is generally singular at $E = V(\mathbf{x})$. Requiring $\Phi(\mathbf{x})$ to be regular at this point makes the energy

² This important property is bypassed in most modern textbooks. See also Ref. [12].

³ But not with perturbative Coulomb photon/gluon exchange, which (for finite fermion masses) transmits energy as well as 3-momentum. See footnote 1.

spectrum discrete rather than continuous as in the Dirac case. The $f\bar{f}$ states in $D = 1 + 1$ were found to transform correctly under boosts [1], and the energy eigenvalues of (6) satisfy $E = \sqrt{\mathbf{P}^2 + M^2}$ [8]. Fig. 1(b) shows one component of the 2×2 wave function $\Phi(x)$ for $m/e = 4$, compared to the Schrödinger wave function with the reduced mass $m/e = 2$. The comparison is qualitatively similar to the Dirac case in Fig. 1(a).

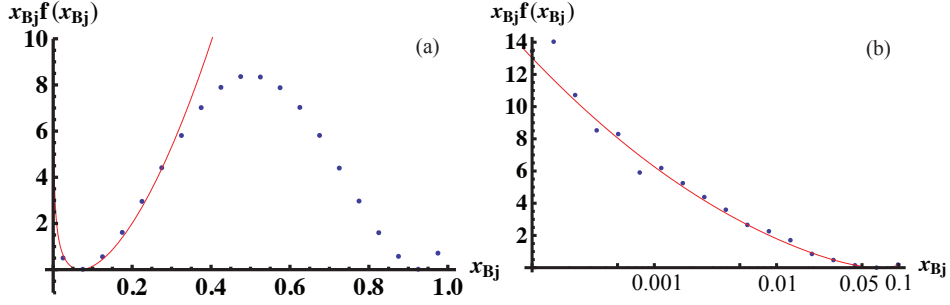


Fig. 2. (a) $f\bar{f}$ ground state parton distribution in $D = 1 + 1$ for $m/e = 0.1$ (preliminary, [1]). (b) The same distribution on a logarithmic scale. The dots are numerical results and the curve shows an analytic approximation valid at low x_{Bj} .

3. Form factors and parton distributions

The matrix element of the electromagnetic current $j^\mu(z) = \bar{\psi}(z)\gamma^\mu\psi(z)$ between $f\bar{f}$ bound states (5) gives the form factor [1],

$$\begin{aligned} F_{AB}^\mu(z) &\equiv \langle B(P_b) | j^\mu(z) | A(P_a) \rangle \\ &= e^{i(P_b - P_a) \cdot z} \int d\mathbf{x} e^{i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x} / 2} \text{Tr} [\Phi_B^\dagger(\mathbf{x}) \gamma^\mu \gamma^0 \Phi_A(\mathbf{x})] \end{aligned} \quad (7)$$

Gauge invariance, $\partial_\mu F_{AB}^\mu(z) = 0$, holds as a consequence of the bound state equation (6) satisfied by the wave functions Φ_A, Φ_B .

The quark distribution of target state A is obtained in the Bjorken limit where the photon virtuality and the mass of the final state B tend to infinity. Since all states have zero width (before the perturbative corrections) an averaging procedure needs to be applied. The relative normalization of the wave functions Φ_B can be determined from duality between the contributions of bound states and free quarks to the imaginary parts of current propagators. Our preliminary result for the quark distribution of a relativistic ground state ($m/e = 0.1$) in $D = 1 + 1$ is shown in Fig. 2. The rise of the distribution at low x_{Bj} is attributed to $f\bar{f}$ pairs, indicating again the inclusive nature of the wave functions obtained with retarded boundary conditions.

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